Optimization via Surjective Factorizations

Gustavo de Sousa

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Abstract

We study the transfer of optimization problems across surjective factorizations of maps. Working in the general setting of posets, we show that extrema of a function J can be equivalently expressed after reparametrization through a surjective map and a section. We also give a factorization form where J decomposes as $H \circ \Phi$ with Φ surjective. These theorems show that optimization can often be reduced to a smaller or simpler search domain without altering the extremal values.

1 General Setting

Let $(X, \leq)$ be a partially ordered set. For any subset $A \subseteq X$, we write

$$\inf A, \quad \sup A$$

for the infimum and supremum in X, when they exist. When X is a lattice, these always exist; when X is totally ordered, they coincide with minima and maxima.

Given any function $J: U \to X$ from a set U into a poset, we are interested in computing $\inf J(U)$ and $\sup J(U)$. The following theorems describe when this task can be transferred to another set Y via surjective maps.

2 Main Theorems

Theorem 1 (Transfer via section)**.** Let U, X, Y be sets with $(X, \leq)$ a poset. Suppose we have

$$J: U \to X, \quad G: X \to Y, \quad S: Y \to X,$$

satisfying:

(i) $G \circ J: U \to Y$ is surjective,

(ii) $S \circ G = \mathrm{id}_X$.

Then, whenever the infimum and supremum exist,

$$\inf_{u\in U} J(u) = \inf_{y\in Y} S(y), \qquad \sup_{u\in U} J(u) = \sup_{y\in Y} S(y).$$

Proof. For each $u \in U$, $S(G(J(u))) = J(u)$. Since $G\circ J$ is surjective, every $y \in Y$ has the form $y = G(J(u))$ for some u. Thus $\{J(u) : u \in U\} = \{S(y) : y \in Y\}$. Taking infima and suprema yields the result. □

Theorem 2 (Factorization form)**.** Let U and Y be sets, $(X, \leq)$ a poset.

Suppose there exists a surjective map $\Phi\colon U \to Y$ and a map $H\colon Y \to X$ such that

$$J = H \circ \Phi.$$

Then

$$\inf_{u\in U} J(u) = \inf_{y\in Y} H(y), \qquad \sup_{u\in U} J(u) = \sup_{y\in Y} H(y).$$

Proof. By construction, $J(U) = H(\Phi(U))$. Since Φ is surjective, $\Phi(U) = Y$, so $J(U) = H(Y)$. Taking infima and suprema over these sets gives the claim. □

Remark 1. Theorem 1 and Theorem 2 are equivalent: a section S as in Theorem 1 gives the factorization in Theorem 2 by setting $\Phi = G \circ J$ and $H = S$. Conversely, a factorization as in Theorem 2 can be encoded in the form of Theorem 1 by taking Y as the codomain of Φ and defining G and S accordingly.

3 Examples

Example 1 (Binning in optimization)**.** Let U be a finite set and $J\colon U \to \mathbb{R}$ a cost function. Fix a bin width $\Delta > 0$ and define

$$G(c) = \left\lfloor \tfrac{c}{\Delta} \right\rfloor, \qquad c \in \mathbb{R}.$$

This partitions $\mathbb{R}$ into bins indexed by integers.

For each bin y, define a representative

$$S(y) = \operatorname*{arg\,min}_{\substack{u\in U\\ G(J(u))=y}} J(u).$$

Then

$$\min_{u\in U} J(u) = \min_{y\in Y} J(S(y)).$$

This shows that optimization can be performed by searching one representative per bin.

Example 2 (Quotienting symmetries)**.** Suppose a group G acts on U and the cost function $J\colon U \to \mathbb{R}$ is invariant under the action. Then J factors through the quotient map $\pi\colon U \to U/G$. Writing $H([u]) = J(u)$, we obtain

$$\min_{u\in U} J(u) = \min_{[u]\in U/G} H([u]),$$

so the optimization can be reduced to orbit representatives.

4 Conclusion

We have shown that optimization problems can be transferred across surjective factorizations without changing the extremal values. This principle holds in the general setting of posets, unifying several practical strategies such as binning, quotienting by symmetries, or reparametrization. In applications, this allows optimization to be performed in smaller or more structured domains while guaranteeing correctness of extremal values.