Measures Generated by Affine Covers and an Integral via Subgraphs

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Abstract

We formalize a short self-contained construction. Start with an arbitrary set function $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ and define the inner and outer pre-measures by infima over supersets and suprema over subsets. We then give a generation method for inner/outer measures using copies of a template set $M \subset \mathbb{R}^n$ acted on by a family of C^1 affine maps. Finally we define the integral of a nonnegative function as the measure of its subgraph (using the product with Lebesgue measure on the vertical axis), and extend the definition to signed functions by decomposition into positive and negative parts. Basic properties are proved and examples are given.

1 Notation and standing assumptions

Let $n \in \mathbb{N}$ and write $\mathcal{P}(\mathbb{R}^n)$ for the power set of $\mathbb{R}^n$. Let

$$\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0, \infty]$$

be any set function. We do not assume measurability or countable additivity of μ a priori. For a set $S \subset \mathbb{R}^n$ we write

$$\mu_*(S) := \inf\{\mu(U) \mid S \subseteq U \subset \mathbb{R}^n\}$$

and

$$\mu^*(S) := \sup\{\mu(V) \mid V \subseteq S \subset \mathbb{R}^n\}.$$

The notation follows the convention that μ_* is an *outer*-type quantity (infimum over covers) and μ^* is an *inner*-type quantity (supremum over contained sets). When μ is itself a measure on a sigma-algebra these become the usual inner and outer measures restricted to $\mathcal{P}(\mathbb{R}^n)$.

2 Generation by template sets and differentiable maps

Fix a bounded template set $M \subset \mathbb{R}^n$ with $0 < \mu(M) < \infty$ and a countable family $\mathcal{A}$ of C^1 injective maps $A : \mathbb{R}^n \to \mathbb{R}^n$ such that each A is a diffeomorphism onto its image and the derivative $DA(x)$ is constant when A is affine (for general C^1 maps we use $|\det DA(x)|$ below). For an affine map the Jacobian determinant is independent of x; write $\det DA$ for it.

Definition 2.1 (Generated outer and inner measures)**.** For $S \subset \mathbb{R}^n$ define

$$\tilde{\mu}_*(S) := \inf\left\{\mu(M) \sum_{A\in\mathcal{A}'} |\det DA| \;\middle|\; S \subseteq \bigcup_{A\in\mathcal{A}'} A(M),\ \mathcal{A}' \subseteq \mathcal{A} \text{ finite}\right\},$$

$$\tilde{\mu}^*(S) := \sup\left\{\mu(M) \sum_{A\in\mathcal{A}'} |\det DA| \;\middle|\; \bigcup_{A\in\mathcal{A}'} A(M) \subseteq S,\ \mathcal{A}' \subseteq \mathcal{A} \text{ finite}\right\}.$$

Remark 2.2. The quantities above are well defined in $[0, \infty]$. They attempt to capture a volume-like quantity obtained by covering S with finitely many distorted copies of a template M and weighting each copy by the local linear expansion $|\det DA|$. If μ is translation invariant and A are linear maps then this construction coincides with the familiar scaling behaviour of Lebesgue measure.

Proposition 2.3 (Basic monotonicity and subadditivity)**.** *The set functions $\tilde{\mu}_*$ and $\tilde{\mu}^*$ are monotone in the sense that $S \subset T \Rightarrow \tilde{\mu}_*(S) \leq \tilde{\mu}_*(T)$ and $S \subset T \Rightarrow \tilde{\mu}^*(S) \leq \tilde{\mu}^*(T)$. Moreover $\tilde{\mu}_*$ is finitely subadditive.*

Proof. Monotonicity is immediate from the definitions since enlarging the target in the infimum can only increase the admissible covering families and enlarging the target in the supremum can only decrease the class of admissible finite unions contained in it. For finite subadditivity let $S, T \subset \mathbb{R}^n$ and let > 0. Choose finite subfamilies $\mathcal{A}_S, \mathcal{A}_T$ such that $S \subset \bigcup_{A \in \mathcal{A}_S} A(M)$, $T \subset \bigcup_{A \in \mathcal{A}_T} A(M)$ and

$$\mu(M) \sum_{A \in \mathcal{A}_S} |\det DA| \leq \tilde{\mu}_*(S)+, \qquad \mu(M) \sum_{A \in \mathcal{A}_T} |\det DA| \leq \tilde{\mu}_*(T)+.$$

Then $S \cup T$ is covered by $\mathcal{A}_S \cup \mathcal{A}_T$ and the corresponding cost is the sum of the two costs. Taking $\to 0$ gives finite subadditivity. □

3 Integral via subgraph measure

To define an integral we require a product measure on $\mathbb{R}^{n+1}$. Let λ denote Lebesgue measure on $\mathbb{R}$. Formally extend μ to a set function on $\mathbb{R}^{n+1}$ by setting for $B \subset \mathbb{R}^n$ and interval $I \subset \mathbb{R}$

$$(\mu \times \lambda)(B \times I) := \mu(B)\,\lambda(I)$$

and declaring $(\mu \times \lambda)$ on arbitrary rectangles by finite additivity, then extend to an outer measure by the usual Carathéodory outer measure procedure if needed. For the purposes of this note it suffices to observe that the subgraph of a nonnegative function is a measurable set for reasonable choices of μ and the product.

Definition 3.1 (Integral via subgraph)**.** Let $S \subset \mathbb{R}^n$ and let $f : S \to [0, \infty]$ be a nonnegative function. Define

$$\int_S f\,d\mu \ := \ (\mu \times \lambda)\big(\{(x, y) \in S \times [0, \infty) : \ 0 \leq y \leq f(x)\}\big). \tag{1}$$

For an arbitrary real-valued function f define the positive and negative parts

$$f^+(x) := \max\{f(x), 0\}, \qquad f^-(x) := \max\{-f(x), 0\}$$

and set

$$\int_S f\,d\mu := \int_S f^+\,d\mu - \int_S f^-\,d\mu$$

whenever at least one of the two integrals on the right is finite.

Proposition 3.2 (Linearity on simple combinations and monotonicity)**.** *If $f, g : S \to [0, \infty]$ then $f \leq g$ implies $\int_S f\,d\mu \leq \int_S g\,d\mu$. If $\alpha \geq 0$ then $\int_S (\alpha f)\,d\mu = \alpha \int_S f\,d\mu$. If f, g are simple functions with disjoint supports then $\int_S (f + g)\,d\mu = \int_S f\,d\mu + \int_S g\,d\mu$.*

Proof. Monotonicity follows from inclusion of subgraphs. Homogeneity in α follows because scaling the vertical variable by α multiplies the Lebesgue measure of vertical slices by α. For simple functions write each as a finite linear combination of indicator functions of measurable sets and reduce to finite additivity of the product measure on rectangles. □

Remark 3.3 (Relation with classical Lebesgue integral). If μ is Lebesgue measure on $\mathbb{R}^n$ then the construction in Equation (1) coincides with the standard Lebesgue integral for nonnegative functions. The usual theorems such as the Monotone Convergence Theorem and Fubini's theorem follow from the standard measure-theoretic development once the product measure is constructed and the function is shown to be measurable.

4 Examples

1. If $\mu = \mathcal{L}^n$ is Lebesgue measure and $\mathcal{A}$ is the family of linear homotheties $A(x) = rx + t$ with $r > 0$ and $t \in \mathbb{R}^n$ then

$$\tilde{\mu}_*(S) \qquad \text{and} \qquad \tilde{\mu}^*(S)$$

are compatible with scaling by factors r^n and recover expected behaviour for sets that can be tiled by scaled copies of M.

2. If μ is counting measure on a discrete subset of $\mathbb{R}^n$ then integrals defined by subgraphs reduce to sums of vertical measures over points.

5 Concluding remarks

The constructions above show how minimal assumptions yield a useful integral and a family of inner/outer pre-measures. To turn the pre-measures into full measures and to recover the whole toolbox of measure theory one must perform the Carathéodory outer measure extension, define measurable sets and verify countable additivity on the resulting sigma-algebra. When μ already possesses invariances and regularity the generated quantities improve and in many cases coincide with classical objects.

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References

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[2] G. B. Folland, *Real Analysis*, 2nd ed., Wiley, 1999.