Closed Systems: a First-Order Formalization

Abstract

A compact first-order presentation of *closed systems*. Two-sorted signature. Functions are first-class objects. Composition and application axioms are stated. Forward-closure, orbits, generating sets and basic algebraic facts are given with short proofs. Remarks explain how to encode the generated monoid inside FOL.

1 Overview and conventions

We work in many-sorted first-order logic with two sorts: Elem (elements) and Func (unary endomaps of Elem). Application and composition are primitive. Predicates of the form $S(x)$ mean "x belongs to the definable subset $S \subseteq \mathsf{Elem}$". Notationally we sometimes write $f \circ x$ or $f(x)$ for application. The family Φ is a distinguished set of generators; $\Phi(f)$ is the predicate "$f \in \Phi$".

2 Signature and axioms

Definition 2.1 (Signature)**.** *The language contains:*

- *sorts* Elem *and* Func*,*
- *an application operator* $\mathsf{App} : \mathsf{Func} \times \mathsf{Elem} \to \mathsf{Elem}$*,*
- *a composition operator* $\mathsf{Comp} : \mathsf{Func} \times \mathsf{Func} \to \mathsf{Func}$*,*
- *a constant* $\mathsf{id} \in \mathsf{Func}$*.*

Definition 2.2 (Axioms)**.** *The following axioms hold:*

$$\begin{aligned}
&\mathsf{App}(\mathsf{id}, x) = x,\\
&\mathsf{App}(\mathsf{Comp}(f, g), x) = \mathsf{App}(f, \mathsf{App}(g, x)),\\
&\mathsf{Comp}(f, \mathsf{id}) = f = \mathsf{Comp}(\mathsf{id}, f),\\
&\mathsf{Comp}(f, \mathsf{Comp}(g, h)) = \mathsf{Comp}(\mathsf{Comp}(f, g), h).
\end{aligned}$$

Thus, $(\mathsf{Func}, \mathsf{Comp}, \mathsf{id})$ *forms a monoid acting on* Elem.

3 Forward-closure and orbits

Definition 3.1. *For a unary predicate* $S(\cdot)$ *and a function variable* $\phi \in \mathsf{Func}$ *define*

$$\mathbf{Fch}_\phi(S) \ :\Longleftrightarrow \ \forall x \in \mathsf{Elem}\ \big(S(x) \wedge \Phi(\phi) \Rightarrow S(\phi \circ x)\big).$$

Write $\mathbf{Fch}_\Phi(S)$ *for* $\forall \phi \in \mathsf{Func}\ (\Phi(\phi) \Rightarrow \mathbf{Fch}_\phi(S))$.

Definition 3.2. *Informally the monoid generated by* Φ *is*

$$\langle \Phi \rangle \ = \ \{\, \phi_n \circ \ \ldots \ \circ \phi_1 : n \geq 0,\ \phi_i \in \Phi \,\},$$

with the convention $n = 0$ *yields* id. *For* $x \in \mathsf{Elem}$ *the forward-orbit is*

$$\mathcal{O}(x) := \{\psi \circ x : \psi \in \langle \Phi \rangle\}.$$

Proposition 3.1 (Orbit characterization)**.** $\mathbf{Fch}_\Phi(S)$ *holds iff for every* x *with* $S(x)$ *we have* $\mathcal{O}(x) \subseteq S$.

Proof. If $\mathbf{Fch}_\Phi(S)$ then every generator sends points of S back to S. By induction on word length and action compatibility (A3) every element of $\langle \Phi \rangle$ preserves S. Conversely, if each orbit of $x \in S$ lies in S then in particular images by generators lie in S, so $\mathbf{Fch}_\Phi(S)$. □

4 Set operations

Here, we treat definable subsets as predicates. Propositions like $A \cup B$, $A \cap B$, and $A \subseteq B$ are defined as, respectively $\forall x \in \mathsf{Elem} : A(x) \vee B(x)$, $\forall x \in \mathsf{Elem} : A(x) \wedge B(x)$, and $\forall x \in \mathsf{Elem} : A(x) \implies B(x)$.

Proposition 4.1. *If* $\mathbf{Fch}_\Phi(A)$ *and* $\mathbf{Fch}_\Phi(B)$ *then,*
$\mathbf{Fch}_\Phi(A \cup B)$ *and* $\mathbf{Fch}_\Phi(A \cap B)$.

Proof. Union: let x satisfy $A(x) \vee B(x)$. For any $\phi \in \Phi$ apply closure in the piece containing x. Intersection: let x satisfy $A(x) \wedge B(x)$; apply both closures. □

Converse failure. The converses need not hold. Example: $\mathsf{Elem} = \{a, b\}$, $\mathsf{Func} = \{\mathsf{id}, \phi\}$ with $\phi \circ a = b$, $\phi \circ b = b$, and $\Phi(\phi)$ true. Put $A = \{a\}$, $B = \{b\}$. Then $A \cup B$ is closed but A is not.

5 Closure operator and generators

Definition 5.1. *For $T \subseteq \mathsf{Elem}$ define*

$$\mathrm{Fcl}_\Phi(T) := \{\psi \circ x : x \in T,\ \psi \in \langle\Phi\rangle\}.$$

Proposition 5.1. $\mathrm{Fcl}_\Phi : \mathcal{P}(\mathsf{Elem}) \to \mathcal{P}(\mathsf{Elem})$ *is a closure operator: it is extensive, monotone and idempotent.*

Proof. Extensive: $\mathsf{id} \circ x = x$. Monotone: if $T \subseteq U$ then orbits from T are in those from U. Idempotent: $\langle\Phi\rangle$ is a monoid so further closure adds nothing new. □

Definition 5.2 (Generating set / minimal basis)**.** *A set $B \subseteq S$ generates forward-closed S when $\mathrm{Fcl}_\Phi(B) = S$. A minimal generating set meets each orbit contained in S in at least one representative. Choosing one representative per orbit yields a (generally non-unique) minimal generating set.*

6 Morphisms and equivariance

Definition 6.1. *Given systems (X, Φ) and (Y, Ψ), a map $f : X \to Y$ is a* morphism *if there exists $\widehat{f} : \mathsf{Func}_X \to \mathsf{Func}_Y$ such that*

$$\forall g \in \mathsf{Func}_X,\ \forall x \in X : \quad f(g \circ x) = \widehat{f}(g) \circ f(x).$$

If f and $\widehat{f}$ are bijections and the relation holds both ways then f is a homeomorphism analogue.

Proposition 6.1. *Morphisms send forward-closed sets to forward-closed sets; preimages are forward-closed under the dual condition.*

Proof. Standard equivariance argument. Fix $A \subseteq X$ closed. If $y = f(x)$ then for any generator in the target find the associating source map and use A closedness. □

7 Examples and short remarks

- If $\Phi = \{\mathsf{id}\}$ every subset is forward-closed.
- If Φ is a single shift on $\mathbb{Z}$ or $\mathbb{N}$ orbits are rays and minimal generators correspond to left endpoints.
- Minimal generating sets exist by picking one representative per orbit but are not unique.